JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **188**, 855–866 (1994)

The Three-Term Recurrence Relation and the Differentiation Formulas for Hypergeometric-type Functions

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Submitted by William F. Ames

Received February 8, 1993

The functions of hypergeometric type are the solutions $y \equiv y_\nu(x)$ of the differential equation $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$ where σ, τ are polynomials of degrees not higher than 2 and 1, respectively and λ is a constant. Here we consider a class of functions of hypergeometric type with the additional condition that $\lambda + \nu\tau' + \frac{1}{2}\nu(\nu - 1)\sigma'' = 0$, ν being a complex number, in general. Moreover, we assume that the coefficients of the polynomials σ and τ have no dependence on ν. To this class of functions belong Gauss, Kummer, and Hermite functions, the classical orthogonal polynomials, and many other functions encountered in linear and non-linear physics. We obtain two important structural properties of these functions: (i) the so-called three-term recurrence relation which correlates three functions of successive orders, and (ii) the differentiation formulas (also called ladder or structure relations or, even, differential-recurrence relations) which relate the first derivative $y_\nu'(z)$ with the functions $y_\nu(z)$ and $y_{\nu+1}(z)$ or $y_{\nu-1}(z)$. Finally, these three relationships are applied to the polynomials of hypergeometric type which form a broad subclass of functions y_ν, where ν is a positive integer number and the associated contour is closed. For completeness, the explicit expressions corresponding to all classical orthogonal polynomials (Jacobi, Laguerre, Hermite, and Bessel) are tabulated. © 1994 Academic Press, Inc.

I. INTRODUCTION

A long-standing problem, not yet solved, in the theory of special functions [1, 2, 5] whose solutions would be very helpful in applied mathemat-

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0022-247X/94 $6.00

409/188/3-10

ics as well as in many quantum-mechanical problems of physics [4], is the determination of the differentiation formulas (also called ladder relationships or, even, structure relations) of the hypergeometric-type polynomials $\{y_n(x)\}$ only from the coefficients of the differential equation

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0 \tag{1}$$

which is fulfilled by those polynomials. The coefficients $\sigma(z)$, $\tau(z)$, and λ are themselves polynomials of degrees not higher than 2, 1 and 0 respectively.

This problem was partially solved by Tricomi [5, pp. 212–215] in the sense that he was able to calculate the coefficients $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ of the differentiation formula

$$\sigma(z)y_n'(z) = (\tilde{\alpha}_n z + \tilde{\beta}_n)y_n(z) + \tilde{\gamma}_n y_{n-1}(z) \tag{2}$$

but not the coefficient $\tilde{\gamma}_n$. To evaluate $\tilde{\gamma}_n$ he needed to know not only Eq. (1) but also the three-term recurrence relation which involves $y_{\nu+1}(z)$, $y_\nu(z)$, and $y_{\nu-1}(z)$. We should say immediately that, of course, the specific differentiation formulas of the four families of classical orthogonal polynomials (Jacobi, Laguerre, Hermite, and Bessel) are known [1, 2]. Apparently no results other than those of Tricomi have been found up to now (see, e.g. pp. 148–149 of [1] and pp. 166–167 of [2]).

In this paper we solve not only an extended version of the aforementioned problem in the sense that we calculate the coefficients of both the differentiation formulas and the three-term recurrence relations directly in terms of (σ, τ, λ) but we pose and solve this problem by very elementary means in a much more general framework, namely, for the functions of hypergeometric type [4]. These objects are defined as the solution of the equation of hypergeometric type, Eq. (1), of the form [4]

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s - z)^{\nu+1}}\, ds, \tag{3}$$

where

- C_ν is a normalization constant;
- $\rho(z)$ is a solution of the equation

$$(\sigma\rho)' = \tau\rho; \tag{4}$$

- ν is a root of the equation

$$\lambda + \nu\tau' + \frac{1}{2}\nu(\nu - 1)\sigma'' = 0; \tag{5}$$

• C is a contour whose endpoints s_1 and s_2 satisfy the condition

$$\left.\frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\right|_{s_1}^{s_2} = 0; \tag{6}$$

• and the polynomials $\sigma(z)$ and $\tau(z)$ are such that their coefficients have no dependence on ν.

Members of the class of hypergeometric-type functions defined by Eqs. (1)–(6) are, e.g., the Gauss, Kummer, and Hermite functions, all the classical orthogonal polynomials, and many other functions encountered in different fields of mathematics and physics [1, 4].

Many different properties of some specific members of this class are known [1, 2, 4] such as, e.g., the Rodrigues formula, recurrence relations, differentiation formulas, parity, and asymptotic properties and inequalities. However, not so much has been shown for the global class. Here we fix our attention on the following two general properties of the class: a recurrence relation among three functions of successive orders and a differentiation formula which allows us to express the first derivative of a function $y_\nu'(x)$, by means of $y_\nu(x)$ and $y_{\nu-1}(x)$ or $y_{\nu+1}(x)$. The coefficients involved in these relationships will be given explicitly in terms of the normalization constant C_ν and the coefficients of the differential equation (1) which characterize the functions of the above-mentioned class.

The structure of this paper is as follows. In Section II the three-term recurrence relation is discussed in detail and in Section III two differentiation formulas are given and proved, although one of them was previously found. Finally, in Section IV, these three relations are applied to the polynomials of hypergeometric type which are a broad subclass of functions of hypergeometric type $y_\nu(z)$ where ν is a positive integer number n and the associated contour C is a closed one. For completeness, the specific relations of all the classical orthogonal polynomials are tabulated.

II. The Three-Term Recurrence Relation

Here we will show that the functions $y_\nu(z)$ satisfy the recurrence relation

$$A_1 y_{\nu+1}(z) + A_2(z) y_\nu(z) + A_3 y_{\nu-1}(z) = 0, \tag{7}$$

where the coefficients A_1 and A_3 are constants given by

$$A_1 = 2(\nu+1)\tau_{\nu-1}'\tau_{(\nu-1)/2}' \frac{C_\nu}{C_{\nu+1}} \tag{8}$$

$$A_3 = \tau_\nu' \frac{C_\nu}{C_{\nu-1}} [-2\tau_{\nu-1}'^2\sigma + 2\tau_{\nu-1}\tau_{\nu-1}'\sigma' - \tau_{\nu-1}^2\sigma'']$$

$$= \tau_\nu' \frac{C_\nu}{C_{\nu-1}} \{2\tau'[\sigma'(0)\tau_{\nu-1}(0) - \sigma(0)\tau'] \tag{9}$$

$$-\sigma''[\tau_{\nu-1}(0)\tau_{1-\nu}(0) + 4(\nu - 1)\sigma(0)\tau_{(\nu-1)/2}']\}$$

and $A_2(z)$ is a polynomial of degree 1 as

$$\begin{aligned} A_2(z) &= 2\tau_{\nu-1/2}'[-\tau'\tau_\nu - \nu\tau_{\nu-1}'\sigma' + \tau\sigma''] \\ &= -2\tau_{\nu-1/2}'[\tau_\nu'\tau_{\nu-1}'z + \tau_{2\nu}(0)\tau' - \tau_{\nu(1-\nu)}(0)\sigma''], \end{aligned} \tag{10}$$

where we have used the notation

$$\tau_\mu(z) = \tau(z) + \mu\sigma'(z). \tag{11}$$

Alternatively, Eqs. (7)–(10) may be expressed in the following equivalent form,

$$zy_\nu(z) = \alpha_\nu y_{\nu+1}(z) + \beta_\nu y_\nu(z) + \gamma_\nu y_{\nu-1}(z) \tag{12}$$

with

$$\begin{aligned} \alpha_\nu &= \frac{C_\nu}{C_{\nu+1}} (\nu + 1) \frac{\tau_{(\nu-1)/2}'}{\tau_{\nu-1/2}'\tau_\nu'} \\ \beta_\nu &= -\frac{1}{\tau_\nu'\tau_{\nu-1}'} [\tau'\tau_{2\nu}(0) - \sigma''\tau_{\nu(1-\nu)}(0)] \\ \gamma_\nu &= \frac{C_\nu}{C_{\nu-1}} \frac{1}{2\tau_{\nu-1/2}'\tau_{\nu-1}'} \{2\tau'[\sigma'(0)\tau_{\nu-1}(0) - \sigma(0)\tau'] \\ &\quad -\sigma''[\tau_{\nu-1}(0)\tau_{1-\nu}(0) + 4(\nu - 1)\sigma(0)\tau_{(\nu-1)/2}']\}. \end{aligned} \tag{13}$$

Let us now prove the main recurrence relation (7) with coefficients given by Eqs. (8)–(10). We will do it following some ideas developed in [4] for functions of type (3). We start with the summation

$$S = A_1(z)y_{\nu+1}(z) + A_2(z)y_\nu(z) + A_3(z)y_{\nu-1}(z), \tag{14}$$

where A_i, $i = 1, 2, 3$, are arbitrary functions of z. The three functions of hypergeometric type involved in this summation may be expressed as

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu+1}}\, ds = \frac{C_\nu}{\rho(z)} \frac{1}{\nu} \int_C \frac{\tau_{\nu-1}(s)\sigma^{\nu-1}(s)\rho(s)}{(s-z)^\nu}\, ds \tag{15}$$

$$\begin{aligned} y_{\nu+1}(z) &= \frac{C_{\nu+1}}{\rho(z)} \int_C \frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\, ds \\ &= \frac{C_{\nu+1}}{\rho(z)} \frac{1}{\nu(\nu+1)} \int_C \frac{\sigma^{\nu-1}(s)[\tau_\nu'\sigma(s) + \tau_\nu(s)\tau_{\nu-1}(s)]\rho(s)}{(s-z)^\nu}\, ds \end{aligned} \tag{16}$$

$$y_{\nu-1}(z) = \frac{C_{\nu-1}}{\rho(z)} \int_C \frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^\nu}\, ds, \tag{17}$$

where we have used Eq. (3) in writing the equality of these three expressions, and then for the second equality one, in Eq. (15), and two, in Eq. (16), integrations by parts have been performed. In doing so, we have made use of condition (6) and the equation

$$[\sigma^\nu(s)\rho(s)]' = \sigma^{\nu-1}(s)\tau_{\nu-1}(s)\rho(s)$$

which is a consequence of (4) and (11).

Now we replace expressions (15)–(17) in Eq. (14). We obtain that

$$S = \frac{1}{\rho(z)} \int_C \frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^\nu} P(s)\, ds \tag{18}$$

with

$$P(s) := \frac{A_1(z)C_{\nu+1}}{\nu(\nu+1)} [\tau_\nu'\sigma(s) + \tau_\nu(s)\tau_{\nu-1}(s)] + \frac{A_2(z)C_\nu}{\nu} \tau_{\nu-1}(s) + A_3(z)C_{\nu-1}. \tag{19}$$

Let us choose the coefficients $A_i(z)$ so that

$$\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^\nu} P(s) = \frac{d}{ds}\left[\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-1}} Q(s)\right], \tag{20}$$

where $Q(s)$ is a polynomial. Then, Eq. (18) transforms into

$$S = \frac{1}{\rho(z)} \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-1}} Q(s)\Big|_{s_1}^{s_2} = 0 \tag{21}$$

The sum vanishes providing that the contour C is such that

$$\frac{\sigma^{\nu}(s)\rho(s)}{(s-z)^{\nu-1}} s^m \Big|_{s_1}^{s_2} = 0, \qquad \text{for each } m = 0, 1, 2, 3, \ldots \tag{22}$$

which is a condition similar to (6).

Equation (20) gives, after the derivation contained in it, that

$$P(s) = [\tau_{\nu-1}(s)(s-z) - (\nu-1)\sigma(s)]Q(s) + \sigma(s)(s-z)Q'(s). \tag{23}$$

Then, expressions (19) and (23) of $P(s)$ are identically equal. Notice that it is enough that $Q(s)$ be a constant; we can take $Q(s) = 1$ without any loss of generality. Then, with the following Taylor's developments around the position $s = z$,

$$\tau_\mu(s) = \tau_\mu(z) + \tau'_\mu(s-z)$$

$$\sigma(s) = \sigma(z) + \sigma'(z)(s-z) + \frac{\sigma''}{2}(s-z)^2$$

and equating the coefficients of the powers of $(s-z)$ on both sides of the identity, we finally obtain a system of three equations with the three unknowns A_i, $i = 1, 2$, and 3. The solution of this system of equations leads in a simple and straightforward manner to the searched values (8), (10), and (9) for the coefficients A_1, A_2, and A_3, respectively.

III. The Differentiation Formulas

The relation among the functions $y_\nu(z)$, $y_{\nu+1}(z)$, or $y_{\nu-1}(z)$ and the first derivative $y'_\nu(z)$ is generally referred to by the names differentiation formula, ladder relation, structure relation, and, even, differential–recurrence relation.

One should immediately say that the relation between $y'_\nu(z)$, $y_\nu(z)$, and $y_{\nu+1}(z)$ is known [4] as

$$\sigma(z)y'_\nu(z) = \frac{\tau'_{(\nu-1)/2}}{\tau'_\nu}\left[(\nu+1)\frac{C_\nu}{C_{\nu+1}} y_{\nu+1}(z) - \tau_\nu(z)y_\nu(z)\right). \tag{24}$$

Since $\tau_\nu(z) = \tau_\nu(0) + \tau'_\nu z$, this equation transforms into

$$\sigma(z)y'_\nu(z) = (\bar{\alpha}_\nu z + \bar{\beta}_\nu)y_\nu(z) + \bar{\gamma}_\nu y_{\nu+1}(z) \tag{25}$$

with

$$\bar{\alpha}_\nu = -\tau'_{(\nu-1)/2}$$
$$\bar{\beta}_\nu = -\tau'_{(\nu-1)/2}\frac{\tau_\nu(0)}{\tau'_\nu} \tag{26}$$
$$\bar{\gamma}_\nu = (\nu + 1)\frac{\tau'_{(\nu-1)/2}}{\tau'_\nu}\frac{C_\nu}{C_{\nu+1}}.$$

The second differentiation formula gives the relation among $y'_\nu(z)$, $y_\nu(z)$, and $y_{\nu-1}(z)$. We can obtain it in two different ways: (i) Eliminating $y_{\nu+1}(z)$ in Eq. (24) by means of the three-term recurrence relation previously found, or (ii) applying the method already discussed in the previous section to the involved functions $y'_\nu(z)$, $y_\nu(z)$, and $y_{\nu-1}(z)$. In the latter way we need the integral representation (3) of $y_\nu(z)$ and that of $y'_\nu(z)$ which is [4]

$$y'_\nu(z) = \frac{\tau'_{(\nu-1)/2}C_\nu}{\sigma(z)\rho(z)}\int_C \frac{\sigma^\nu(s)\rho(s)}{(s - z)^\nu}\,ds.$$

Both ways result in

$$\begin{aligned}\sigma(z)y'_\nu(z) = &-\frac{\nu}{\nu - 1}\left[\tau(z) - \frac{\tau'_{(\nu-1)/2}\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\right] y_\nu(z)\\ &+\frac{C_\nu}{C_{\nu-1}}\left\{\sigma(z)\tau'_{\nu-1} + \frac{\tau_{\nu-1}(z)}{\nu - 1}\left[\tau(z)\right.\right.\\ &\left.\left.-\frac{\tau'_{(\nu-1)/2}}{\tau'_{\nu-1}}\tau_{\nu-1}(z)\right]\right\} y_{\nu-1}(z).\end{aligned} \tag{27}$$

The use of the Taylor's developments for $\sigma(z)$, $\tau(z)$, and $\tau_{\nu-1}(z)$ allows us to rewrite the differentiation formula (27) as

$$\sigma(z)y'_\nu(z) = (\tilde{\alpha}_\nu z + \tilde{\beta}_\nu)y_\nu(z) + \tilde{\gamma}_\nu y_{\nu-1}(z) \tag{28}$$

with

$$\tilde{\alpha}_\nu = \frac{\nu}{2}\sigma''$$
$$\tilde{\beta}_\nu = \nu\left[\sigma'(0) - \frac{\sigma''}{2}\frac{\tau_{\nu-1}(0)}{\tau'_{\nu-1}}\right] \tag{29}$$
$$\tilde{\gamma}_\nu = \frac{C_\nu}{C_{\nu-1}}\left\{\sigma(0)\tau'_{\nu-1} - \tau_{\nu-1}(0)\left[\sigma'(0) - \frac{\sigma''}{2}\frac{\tau_{\nu-1}(0)}{\tau'_{\nu-1}}\right]\right\}.$$

IV. Applications to Polynomials of Hypergeometric Type

The polynomials of hypergeometric type $y_n(z)$ are solutions of Eq. (1) of the form

$$y_n(z) = \frac{C_n}{\rho(z)} \oint \frac{\sigma^n(s)\rho(s)}{(s-z)^{n+1}}\, ds, \tag{30}$$

where C_n is a constant, $\rho(z)$ is a solution of the equation $(\sigma\rho)' = \tau\rho$, and the positive integer number n is a root of the equation

$$\lambda + n\tau' + \frac{1}{2} n(n-1)\sigma'' = 0. \tag{31}$$

They satisfy the formula of Rogrigues type [4],

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \tag{32}$$

where the normalization factor B_n is related to the constant C_n as

$$C_n = \frac{B_n n!}{2\pi i}. \tag{33}$$

For convenience, let us point out that B_n is related to the leading coefficient a_n of the polynomial $y_n(z)$ as [4]

$$a_n = B_n \prod_{k=0}^{n-1} \left[\tau' + \frac{1}{2}(n+k-1)\sigma''\right].$$

Then, for monic polynomials $\hat{y}_n(z)$, i.e., when $a_n = 1$, one has that the normalization factor B_n in the Rodrigues formula is

$$\hat{B}_n = \left\{\prod_{k=0}^{n-1} \left[\tau' + \frac{1}{2}(n+k-1)\sigma''\right]\right\}^{-1} \tag{34}$$

and correspondingly the constant $\hat{C}_n = \hat{B}_n n!/(2\pi i)$.

Now let us obtain the three-term recurrence relation and the differentiation formula for the polynomials $y_n(z)$. The comparison of Eqs. (30)–(31) with Eqs. (3)–(5) shows that, since Eq. (6) is automatically satisfied for any closed contour, it is enough to make the replacement $\nu = n$ in the general expressions calculated in the two previous sections.

(i) *Recurrence Relation*. From (7) and (33) one has

$$y_{n+1}(z) = A_{21} y_n(z) + A_{31} y_{n-1}(z) \tag{35}$$

with

$$\begin{aligned} A_{21} &= \frac{\tau'_{n-1/2}}{\tau'_{n-1}\tau'_{(n-1)/2}} \frac{B_{n+1}}{B_n} [\tau'_n \tau'_{n-1} z + \tau_{2n}(0)\tau' - \tau_{n(1-n)}(0)\sigma''] \\ A_{31} &= \frac{n\tau'_n}{2\tau'_{n-1}\tau'_{(n-1)/2}} \frac{B_{n+1}}{B_{n-1}} \{2\tau'[\sigma(0)\tau' - \sigma'(0)\tau_{n-1}(0)] \\ &\quad + \sigma''[\tau_{n-1}(0)\tau_{1-n}(0) + 4(n-1)\sigma(0)\tau'_{(n-1)/2}]\}. \end{aligned} \tag{36}$$

For monic polynomials $\hat{y}_n(z)$ the normalization factor $B_n = \hat{B}_n$ is given by Eq. (34). Then, the relations (35)–(36) take on the form

$$\hat{y}_{n+1}(z) = (z - \beta_n)\hat{y}_n(z) - \gamma_n \hat{y}_{n-1}(z) \tag{37}$$

with

$$\begin{aligned} \beta_n &= \frac{1}{\tau'_{n-1}\tau'_n} [\sigma''\tau_{n(1-n)}(0) - \tau'\tau_{2n}(0)] \\ \gamma_n &= \frac{n\tau'_{n/2-1}}{2\tau'^2_{n-1}\tau'_{n-3/2}\tau'_{n-1/2}} \{2\tau'[\sigma'(0)\tau_{n-1}(0) - \sigma(0)\tau'] \\ &\quad - \sigma''[\tau_{n-1}(0)\tau_{1-n}(0) + 4(n-1)\sigma(0)\tau'_{(n-1)/2}]\}. \end{aligned} \tag{38}$$

(ii) *Differentiation Formulas*. From Eqs. (25) and (27) one has the following two differentiation formulas for polynomials of hypergeometric type,

$$\begin{aligned} \sigma(z)y'_n(z) &= \frac{\tau'_{(n-1)/2}}{\tau'_n} \left[\frac{B_n}{B_{n+1}} y_{n+1}(z) - \tau_n(z)y_n(z)\right] \\ &= \frac{\tau'_{(n-1)/2}}{\tau'_n} \left\{\frac{B_n}{B_{n+1}} y_{n+1}(z) - [\tau'_n z + \tau_n(0)]y_n(z)\right\} \end{aligned} \tag{39}$$

and

$$\sigma(z)y'_n(z) = (\tilde{\alpha}_n z + \tilde{\beta}_n)y_n(z) + \tilde{\gamma}_n y_{n-1}(z) \tag{40}$$

with

$$\tilde{\alpha}_n = \frac{n}{2}\sigma'', \qquad \tilde{\beta}_n = n\left[\sigma'(0) - \frac{\sigma''}{2}\frac{\tau_{n-1}(0)}{\tau'_{n-1}}\right] \tag{41}$$

$$\tilde{\gamma}_n = \frac{B_n}{B_{n-1}}\, n \left\{\sigma(0)\tau'_{n-1} - \tau_{n-1}(0)\left[\sigma'(0) - \frac{\sigma''}{2}\frac{\tau_{n-1}(0)}{\tau'_{n-1}}\right]\right\}. \tag{42}$$

At this moment one should say that the values (41) for $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ were previously found by [5] and collected in [2]. However, these authors did not find the value (42) for $\tilde{\gamma}_n$; they give the value of $\tilde{\gamma}_n$ in terms of the coefficients of both the differential equation and the three-term recurrence relation (without realizing the correlation between them) of the polynomials.

For monic polynomials, these two differentiation formulas take on the form,

$$\sigma(z)\hat{y}'_n(z) = (\overline{\alpha}_n z + \overline{\beta}_n)\hat{y}_n(z) + \overline{\gamma}_n \hat{y}_{n+1}(z) \tag{43}$$

with

$$\begin{aligned} \overline{\alpha}_n &= -\tau'_{(n-1)/2} \\ \overline{\beta}_n &= -\tau'_{(n-1)/2}\frac{\tau_n(0)}{\tau'_n} \\ \overline{\gamma}_n &= \tau'_{n-1/2} \end{aligned} \tag{44}$$

and

$$\sigma(z)\hat{y}'_n(z) = (\hat{\alpha}_n z + \hat{\beta}_n)\hat{y}_n(z) + \hat{\gamma}_n \hat{y}_{n-1}(z) \tag{45}$$

TABLE I

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$; $\alpha > -1, \beta > -1$	$L_n^{(\alpha)}(x)$; $\alpha > -1$	$H_n(x)$	$B_n^{(\alpha)}(x)$
Γ	$(-1, 1)$	$(0, \infty)$	$(-\infty, \infty)$	Unit disk
$\rho(x)$	$(1-x)^\alpha(1+x)^\beta$	$x^\alpha e^{-x}$	e^{-x^2}	$z^\alpha e^{-2/z}$
$\sigma(x)$	$1 - x^2$	x	1	z^2
$\tau(x)$	$\beta - \alpha - (\alpha + \beta + 2)x$	$1 + \alpha - x$	$-2x$	$(\alpha + 2)z + 2$
λ_n	$n(n + \alpha + \beta + 1)$	n	$2n$	$-n(n + \alpha + 1)$

Note. Basic data of the classical orthogonal polynomials. The domain of orthogonality Γ, the weight function $\rho(x)$ and the coefficients $\sigma(x)$, $\tau(x)$, and λ_n of the second-order differential equation of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the Laguerre polynomials $L_n^{(\alpha)}(x)$, the Hermite polynomials $H_n(x)$, and the Bessel polynomials $B_n^{(\alpha)}(z)$ are shown.

TABLE II

$y_n(x)$	$P_n^{(\alpha,\beta)}(x)$	$L_n^{(\alpha)}(x)$	$H_n(x)$	$B_n^{(\alpha)}(x)$
β_n	$\frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}$	$2n + \alpha + 1$	0	$\frac{-2\alpha}{(2n + \alpha)(2n + \alpha + 2)}$
γ_n	$\frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)}$	$n(n + \alpha)$	$\frac{n}{2}$	$\frac{-4n(n + \alpha)}{(2n + \alpha - 1)(2n + \alpha)^2(2n + \alpha + 1)}$
$\bar{\alpha}_n$	$n + \alpha + \beta + 1$	1	2	$-(n + \alpha + 1)$
$\bar{\beta}_n$	$\frac{(\alpha - \beta)(n + \alpha + \beta + 1)}{2n + \alpha + \beta + 2}$	$-(n + \alpha + 1)$	0	$-\frac{2(n + \alpha + 1)}{2n + \alpha + 2}$
$\bar{\gamma}_n$	$-(2n + \alpha + \beta + 1)$	-1	-2	$2n + \alpha + 1$
$\hat{\alpha}_n$	$-n$	0	0	n
$\hat{\beta}_n$	$\frac{n(\alpha - \beta)}{2n + \alpha + \beta}$	n	0	$\frac{-2n}{2n + \alpha}$
$\hat{\gamma}_n$	$\frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2}$	$n(n + \alpha)$	n	$\frac{4n(n + \alpha)}{(2n + \alpha - 1)(2n + \alpha)^2}$

Note. The coefficients $\{\beta_n, \gamma_n\}$ of the three-term recurrence relation, $\{\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n\}$ of the first differentiation formula and $\{\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\}$ of the second differentiation formula of all the monic classical orthogonal polynomials are given. See Eqs. (38), (44), and (46), respectively, in text for details.

with

$$\hat{\alpha}_n = \tilde{\alpha}_n, \qquad \hat{\beta}_n = \tilde{\beta}_n,$$
$$\hat{\gamma}_n = \frac{\tau'_{n/2-1}}{\tau'_{n-3/2}}\, n \left\{\sigma(0) - \frac{\tau_{n-1}(0)}{\tau'_{n-1}}\left[\sigma'(0) - \frac{\sigma''}{2}\frac{\tau_{n-1}(0)}{\tau'_{n-1}}\right]\right\}. \tag{46}$$

Finally, for the sake of illustration and simultaneously to check our formulas, we apply these relationships to all the orthogonal polynomials of hypergeometric type which are usually called classical orthogonal polynomials, namely,

Jacobi polynomials	$P_n^{(\alpha,\beta)}(x)$,	$\alpha > -1, \beta > -1$
Laguerre polynomials	$L_n^{(\alpha)}(x)$,	$\alpha > -1$
Hermite polynomials	$H_n(x)$	
Bessel polynomials	$B_n^{(\alpha)}(z)$.	

In Table I the basic data of these polynomials are collected; in particular, the domain of orthogonality Γ, the weight function $\rho(x)$, and the coefficients $\sigma(x)$, $\tau(x)$, and λ_n of its differential equation (1). The resulting relations for monic polynomials which have been obtained with our formulas are contained in Table II, where coefficients β_n and γ_n of the recurrence relation (37), the coefficients $\bar{\alpha}_n$, $\bar{\beta}_n$, and $\bar{\gamma}_n$ of the first differentiation formula (43) as well as the coefficients $\hat{\alpha}_n$, $\hat{\beta}_n$, and $\hat{\gamma}_n$ of the second differentiation formula (45) are shown. The values of these coefficients are known although they are disperse in the literature, e.g., the coefficients β_n and γ_n are contained in [3] where they were obtained by a fully different procedure.

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